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THE MATRIX TRANSFORMS THE VECH* TO VECD* OPERATORS

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Abstract. *This article introduces two new matrix operators inspired by the definition of vecd operators, and we call these the vecd***. This operator related to vech** *operators. This operator is constructed the same way as the vec operator, i.e. a matrix of size* $m \times n$ *becomes a column vector, size* $mn \times 1$*. The difference is in the arrangement of entries, which become column vectors. In the vec operator, all entries in the matrix* $m \times n$ *will become entries in the column vector, but the vech and vecd only make some of the entries in the matrix* $m \times n$ *into column vectors with specific rules. We try to define the column vector Also we use the vech** *operator with a different construct from vecd***. We explicitly construct a matrix that transforms* $vech^*(A)$ *to* $vecd^*(A)$*, where* A *is an* $n \times n$ *matrix for* $n \in N$*. We also derive some properties from this transform matrix.*

Kata Kunci: *vec*, *vecd**, *vech**

1. Introduction

The *vec* operator or the *vec* matrix ($mn \times 1$) is a column matrix obtained from a matrix, $m \times n$, i.e., change the matrix into a vector by stacking the column vertically [6]. This definition of the *vec* operator inspired several operators, such as *vech*, *vecd*, and *vecp* (see [7], [12], [13]). Based on these operator definitions, a transformation matrix connects operators with the original matrix and operators with operators. For the *vec* operator and transpose *vec* operator, there is a transformation matrix, known as the commutation matrix [3], and the duplicate matrix that transforms *vec* operator to *vech* operators [15].

For the square matrix, [7] defines *vech* operator in the same way that *vec* is defined, except that for each column of the square matrix only that part of which is on

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or below the diagonal of the square matrix is put into *vech* operator (*vech* =vector-half). For the square matrix, [12] defines the *vecd* operator, i.e., by stacking the entries diagonally and eliminating supra-diagonal entries of the square matrix. Here, d represents the diagonal. This article presents the new operators, based on definition *vecd* and it is called by $vecd^*$. Furthermore, it presents the matrix that transforms $vech^*$ to $vecd^*$ operators for the arbitrary matrix $n \times n$.

The method in this study is a literature study. The first step of this research is to define the $vecd^*$ operator for arbitrary $n \times n$ matrix (see Section 3). Next, define the unique matrix associated with $vech^*$ and $vecd^*$, and find its properties.

2. Basic Theory

Definition 2.1. [10] *Let $A = [a_{ij}]$ be an $m \times n$ matrix, and A_j is the j-th coloumn of A. The $vec(A)$ is the $mn \times 1$ vector given by*

$$vec(A) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}. \tag{2.1}$$

Definition 2.2. [1] [6] *Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $vech(A)$ is the $\dfrac{n(n+1)}{2} \times 1$ column vector that is obtained from $vec(A)$ by eliminating all supra-diagonal elements of A.*

$$vech(A) = \begin{bmatrix} a_{11}\ a_{21}\ \dots\ a_{n1}\ a_{22}\ \dots\ a_{n2}\ \dots\ a_{(n-1)(n-1)}\ a_{n(n-1)}\ a_{nn} \end{bmatrix}^T. \tag{2.2}$$

Definition 2.3. [8] *Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $vech^*(A)$ is the $\dfrac{n(n+1)}{2} \times 1$ column vector that is obtained from $vec(A)$ by eliminating all below main diagonal elements of A.*

$$vech^*(A) = \begin{bmatrix} a_{11}\ a_{12}\ a_{22}\ a_{13}\ a_{23}\ a_{33}\ \dots\ a_{1n}\ a_{2n}\ \dots\ a_{(n-1)n}\ a_{nn} \end{bmatrix}^T. \tag{2.3}$$

Definition 2.4. [12] *Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $vecd(A)$ is the $\dfrac{n(n+1)}{2} \times 1$ column vector that is obtained by stacking main diagonal entry, then diagonal entry below the main diagonal up to entry a_{n1}.*

$$vecd(A) = \begin{bmatrix} a_{11}\ a_{22}\ \dots\ a_{nn}\ a_{21}\ a_{32}\ \dots\ a_{n(n-1)}\ \dots\ a_{(n-1)1}\ a_{n2}\ a_{n1} \end{bmatrix}^T. \tag{2.4}$$

Example 2.5. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then,

- $vec(A) = \begin{bmatrix} a_{11}\ a_{21}\ a_{31}\ a_{12}\ a_{22}\ a_{32}\ a_{13}\ a_{23}\ a_{33} \end{bmatrix}^T$,
- $vech(A) = \begin{bmatrix} a_{11}\ a_{21}\ a_{31}\ a_{22}\ a_{32}\ a_{33} \end{bmatrix}^T$,

- $vech^*(A) = \left[\begin{array}{cccccc} a_{11} & a_{12} & a_{22} & a_{13} & a_{23} & a_{33} \end{array}\right]^T$,
- $vecd(A) = \left[\begin{array}{cccccc} a_{11} & a_{22} & a_{33} & a_{21} & a_{32} & a_{31} \end{array}\right]^T$.

Let S_n denote the set of all permutations of the n element set $[n] = \{1, 2, \ldots, n\}$. A permutation is a one-to-one function from $[n]$ onto $[n]$. If σ is a permutation, we have the identity matrix as follows:

Definition 2.6. [13] *Let σ be a permutation in S_n. Define the permutation matrix $P(\sigma) = [\delta_{i,\sigma(j)}]$, $\delta_{i,\sigma(j)} = ent_{ij}(P(\sigma))$ where*

$$\delta_{i,\sigma(j)} = \begin{cases} 1, & \text{if } i = \sigma(j) \\ 0, & \text{if } i \neq \sigma(j). \end{cases} \tag{2.5}$$

We present definitions of inversion on permutation and elementary product.

Definition 2.7. [2] *Inversion is the occurrence of a larger integer preceding a smaller integer. In comparison, the number of inversions is the total number of integers preceded by a smaller integer in each inversion according to the permutations.*

Definition 2.8. [2] *If the number of inversions of a permutation is an even number, then it is said to be an even permutation, and if it is an odd number, then it is said to be an odd permutation.*

Definition 2.9. [2] *Let A be an $n \times n$ matrix. The elementary product of A is the product of n elements from A without taking elements from the same row or column. In contrast, the signed elementary product of A is the elementary product which is marked $(+1)$ if the permutation is even and (-1) if the permutation is odd.*

Definition 2.10. [2] *An $m \times m$ matrix P whose columns form an orthonormal set of vectors is called an orthogonal matrix. It immediately follows that $P^T P = PP^T = I_m$.*

Theorem 2.11. [2] *Let P be $m \times m$ orthogonal matrix. Then $|P| = \pm 1$, so that P is nonsingular. Consequently, $P^{-1} = P^T$.*

Theorem 2.12. [1] [6] *Every permutation matrix is an orthogonal matrix.*

Definition 2.13. [1] [15] *The Moore-Penrose inverse of the $m \times n$ matrix A is the $n \times m$ matrix, denoted by A^+ which satisfies the conditions*

$$\begin{aligned} AA^+A &= A, \\ A^+AA^+ &= A^+, \\ (AA^+)^T &= AA^+, \\ (A^+A)^T &= A^+A. \end{aligned}$$

Theorem 2.14. [15] *Let A be an $m \times n$ matrix. Then*

$$A^+ = (A^T A)^{-1} A^T \ dan \ A^+ A = I_n \ jika \ rank(A) = n.$$

Definition 2.15. *[15] Let A be an $n \times n$ symmetric matrix. The matrix that transforms $vech(A)$ into $vec(A)$ is called the duplication matrix.*

Definition 2.16. [1][6] *Let an $m \times n$ matrix $A = [a_{ij}]$ and a $p \times q$ matrix $B = [b_{ij}]$, is denoted by the symbol $A \otimes B$ and is defined to be the $mp \times nq$ matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \tag{2.6}$$

obtained by replacing each element a_{ij} of A with the $p \times q$ matrix $a_{ij}B$.

3. Results and Discussion

This paper aims to introduce a new operator like *vech* and *vecd*, and we call the operator with $vecd^*$, and then to present the relationship between $vech^*$ and $vecd^*$.

Definition 3.1. *Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $vecd^*(A)$ is the $\frac{n(n+1)}{2} \times 1$ column vector that is obtained by stacking main diagonal entry, then diagonal entry above the main diagonal up to entry a_{1n}.*

$$vecd^*(A) = \begin{bmatrix} a_{11} \ a_{22} \ \dots \ a_{nn} \ a_{12} \ a_{23} \ \dots \ a_{(n-1)n} \ \dots \ a_{1(n-1)} \ a_{2n} \ a_{1n} \end{bmatrix}^T. \tag{3.1}$$

Based on Example 2.5,

$$vecd^*(A) = \begin{bmatrix} a_{11} \ a_{22} \ a_{33} \ a_{12} \ a_{23} \ a_{13} \end{bmatrix}^T.$$

Example 3.2. Suppose given a matrix A of size 4 as follows:

$$A = \begin{bmatrix} 4 & 1 & -2 & 3 \\ -1 & 5 & -1 & 2 \\ -2 & -1 & 6 & -4 \\ 3 & 2 & -4 & 7 \end{bmatrix}.$$

Then,

$$vech^*(A) = \begin{bmatrix} 4 \ 1 \ 5 \ -2 \ -1 \ 6 \ 3 \ 2 \ -4 \ 7 \end{bmatrix}^T$$

and

$$vecd^*(A) = \begin{bmatrix} 4 \ 5 \ 6 \ 7 \ 1 \ -1 \ -4 \ -2 \ 2 \ 3 \end{bmatrix}^T$$

Let A be an $n \times n$ matrix. In [12], it is stated that there is an $n \times n$ matrix B_n^* that transforms $vech(A)$ to $vecd(A)$, i.e: $B_n^* vech(A) = vecd(A)$. This article will construct a matrix similar to B_n^*, symbolized by $B_n^{*(d)}$, which transforms $vech^*(A)$ to $vecd^*(A)$, where it explicitly assigns each $vech^*(A)$ entry exactly one to the $vecd^*(A)$ entry, i.e.,

$$B_n^{*(d)} vech^*(A) = vecd^*(A). \tag{3.2}$$

Given $B_n^{*(d)}$ for $n = 2, 3, 4, 5$.

$$B_2^{*(d)} = \begin{bmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{bmatrix}, \ B_3^{*(d)} = \begin{bmatrix} 1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,1\,0\,0 \end{bmatrix}, \ B_4^{*(d)} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \end{bmatrix},$$

$$B_5^{*(d)} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \end{bmatrix},$$

Next, the form of $B_n^{*(d)}$, $n = 2, 3, 4, 5$ will be written in a formula. We need several symbols, i.e.:

- $\boldsymbol{e}_{m,n}$ is an $n \times 1$ unit vector for entry 1 in the m-row, and $0'$s elsewhere,
- $O_{m \times n}$ is a zero matrix consisting of m-row and n-column.

Thus form $B_n^{*(d)}$, $n = 2, 3, 4, 5$ can be written as follows:

(a) $$B_2^{*(d)} = \left[\begin{array}{cc} e_{1,2}^T & O_{1\times 1} \\ O_{1\times 2} & e_{1,1}^T \\ \hline O_{1\times 1} & e_{1,2}^T \end{array}\right] = \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \end{bmatrix},$$

(b) $$B_3^{*(d)} = \left[\begin{array}{ccc} e_{1,3}^T & O_{1\times 2} & O_{1\times 1} \\ O_{1\times 2} & e_{1,2}^T & O_{1\times 2} \\ O_{1\times 3} & O_{1\times 2} & e_{1,1}^T \\ \hline O_{1\times 1} & e_{1,3}^T & O_{1\times 2} \\ O_{1\times 2} & O_{1\times 2} & e_{1,2}^T \\ \hline O_{1\times 1} & O_{1\times 2} & e_{1,3}^T \end{array}\right] = \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \\ E_3^{*(d)} \end{bmatrix},$$

$$\text{(c)}\ B_4^{*(d)} = \left[\begin{array}{cccc}
e_{1,4}^T & O_{1\times 3} & O_{1\times 2} & O_{1\times 1} \\
O_{1\times 2} & e_{1,3}^T & O_{1\times 3} & O_{1\times 2} \\
O_{1\times 3} & O_{1\times 2} & e_{1,2}^T & O_{1\times 3} \\
O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,1}^T \\ \hline
O_{1\times 1} & e_{1,4}^T & O_{1\times 3} & O_{1\times 2} \\
O_{1\times 2} & O_{1\times 2} & e_{1,3}^T & O_{1\times 3} \\
O_{1\times 3} & O_{1\times 3} & O_{1\times 2} & e_{1,2}^T \\ \hline
O_{1\times 1} & O_{1\times 2} & e_{1,4}^T & O_{1\times 3} \\
O_{1\times 2} & O_{1\times 3} & O_{1\times 2} & e_{1,3}^T \\ \hline
O_{1\times 1} & O_{1\times 2} & O_{1\times 3} & e_{1,4}^T
\end{array}\right] = \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \\ E_3^{*(d)} \\ E_4^{*(d)} \end{bmatrix},$$

$$\text{(d)}\ B_5^{*(d)} = \left[\begin{array}{ccccc}
e_{1,5}^T & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & O_{1\times 1} \\
O_{1\times 2} & e_{1,4}^T & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} \\
O_{1\times 3} & O_{1\times 2} & e_{1,3}^T & O_{1\times 4} & O_{1\times 3} \\
O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,2}^T & O_{1\times 4} \\
O_{1\times 5} & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,1}^T \\ \hline
O_{1\times 1} & e_{1,5}^T & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} \\
O_{1\times 2} & O_{1\times 2} & e_{1,4}^T & O_{1\times 4} & O_{1\times 3} \\
O_{1\times 3} & O_{1\times 3} & O_{1\times 2} & e_{1,3}^T & O_{1\times 4} \\
O_{1\times 4} & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,2}^T \\ \hline
O_{1\times 1} & O_{1\times 2} & e_{1,5}^T & O_{1\times 4} & O_{1\times 3} \\
O_{1\times 2} & O_{1\times 3} & O_{1\times 2} & e_{1,4}^T & O_{1\times 4} \\
O_{1\times 3} & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,3}^T \\ \hline
O_{1\times 1} & O_{1\times 2} & O_{1\times 3} & e_{1,5}^T & O_{1\times 4} \\
O_{1\times 2} & O_{1\times 3} & O_{1\times 4} & O_{1\times 2} & e_{1,4}^T \\ \hline
O_{1\times 1} & O_{1\times 2} & O_{1\times 3} & O_{1\times 4} & e_{1,5}^T
\end{array}\right] = \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \\ E_3^{*(d)} \\ E_4^{*(d)} \\ E_5^{*(d)} \end{bmatrix},$$

From (a)-(d) it can be seen that the matrix $B_n^{*(d)}$, $n = 2, 3, 4, 5$ forms a pattern for the matrix $B_n^{*(d)}$. Thus, the general form of the $B_n^{*(d)}$ matrix is obtained as follows:

$$
B_n^{*(d)} = \left[\begin{array}{cccccc}
e_{1,n}^T & O_{1\times(n-1)} & O_{1\times(n-2)} & \cdots & \cdots & O_{1\times1} \\
O_{1\times2} & e_{1,(n-1)}^T & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times2} \\
\vdots & O_{1\times2} & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & O_{1\times2} & \ddots & \vdots & \vdots \\
O_{1\times(n-1)} & \vdots & \vdots & O_{1\times2} & e_{1,2}^T & O_{1\times(n-1)} \\
O_{1\times n} & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times2} & e_{1,1}^T \\
\hline
O_{1\times1} & e_{1,n}^T & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times2} \\
O_{1\times2} & O_{1\times2} & e_{1,(n-1)}^T & O_{1\times(n-1)} & \cdots & O_{1\times3} \\
\vdots & \vdots & O_{1\times2} & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & O_{1\times2} & \ddots & O_{1\times(n-1)} \\
O_{1\times(n-1)} & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times2} & e_{1,2}^T \\
\hline
O_{1\times1} & O_{1\times2} & e_{1,n}^T & O_{1\times(n-1)} & \cdots & O_{1\times3} \\
\vdots & O_{1\times3} & O_{1\times2} & e_{1,(n-1)}^T & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & O_{1\times(n-1)} \\
O_{1\times(n-2)} & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times2} & e_{1,3}^T \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
O_{1\times1} & O_{1\times2} & \cdots & O_{1\times(n-2)} & e_{1,n}^T & O_{1\times(n-1)} \\
O_{1\times2} & O_{1\times3} & \cdots & O_{1\times(n-1)} & O_{1\times2} & e_{1,(n-1)}^T \\
\hline
O_{1\times1} & O_{1\times2} & \cdots & \cdots & O_{1\times(n-1)} & e_{1,n}^T
\end{array}\right]
$$

$$
= \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \\ \vdots \\ E_n^{*(d)} \end{bmatrix},
$$

where $E_n^{*(d)}$ is a partition matrix that assigns the diagonal entries of A in $vech^*(A)$ to the same entry in $vecd^*(A)$.

Next, the properties associated with $B_n^{*(d)}$ are as follows:

Theorem 3.3. *The $B_n^{*(d)}$ is a permutation matrix.*

Proof. Based on Definitions 2.6, $B_n^{*(d)}$ is a permutation matrix. □

Corollary 3.4. The $B_n^{*(d)}$ is an orthogonal matrix.

Proof. Based on Theorem 3.3, $B_n^{*(d)}$ is a permutation matrix. Then, based on Theorem 2.12, the $B_n^{*(d)}$ is an orthogonal matrix. □

Theorem 3.5. *Let $B_n^{*(d)}$ be a matrix that transforms $vech^*(A)$ to $vecd^*(A)$. Then*

$$
\det\left(B_n^{*(d)}\right) = \begin{cases} -1, & \text{if } B_n^{*(d)} \text{ is an odd permutation,} \\ 1, & \text{if } B_n^{*(d)} \text{ is an even permutation.} \end{cases}
$$

Proof. Based on Definition 3.3, the entry in a column or row of a permutation matrix is 1, and the other entries are 0, and this matrix comes from an identity matrix in which rows are swapped (an odd or even number). The number of inversions is also related to the number of line swaps. So, the determinant for $B_n^{*(d)}$ can be determined by the number of inversions. By using row reduction, the determination of the matrix $B_n^{*(d)}$ can be seen using odd or even permutations. Based on Definitions 2.8 and 2.9, if $B_n^{*(d)}$, the number of inversions is odd, then the permutation is odd, so that $|B_n^{*(d)}| = -1$ if the number of inversions is even, so the permutation is even, and then $|B_n^{*(d)}| = 1$. The proof is complete. $\square$

Let A be an $n \times n$ symmetric matrix. In [12], it is stated that there is an $n \times n$ matrix D_n that transforms $vech(A)$ to $vec(A)$, i.e: $D_n vech(A) = vec(A)$. Then, we will define a matrix that is analogous to the duplicate matrix which transforms $vecd^*(A)$ to $vec(A)$, symbolized by $D_n^{*(d)}$, where it explicitly assigns each $vecd^*(A)$ entry exactly one to the $vecd(A)$ entry, i.e.,

$$D_n^{*(d)} vecd^*(A) = vecd(A). \tag{3.3}$$

Given $D_n^{*(d)}$ for $n = 2, 3, 4$.

$$D_2^{*(d)} = \begin{bmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,0\,1 \\ 0\,1\,0 \end{bmatrix},\ D_3^{*(d)} = \begin{bmatrix} 1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,1\,0\,0 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,1\,0 \\ 0\,0\,1\,0\,0\,0 \end{bmatrix},\ D_4^{*(d)} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0\,0 \end{bmatrix}.$$

Form $D_n^{*(d)}$, for $n = 2, 3, 4$ can be written as follows:

(a) $$D_2^{*(d)} = \left[\begin{array}{c|c} \left[\, e_1^2 \otimes e_1^{2T} \,\right] & \left[\, e_2^2 \otimes e_1^{1T} \,\right] \\ \hline \left[\, e_2^2 \otimes e_2^{2T} \,\right] & \left[\, e_1^2 \otimes e_1^{1T} \,\right] \end{array}\right],$$

(b) $$D_3^{*(d)} = \left[\begin{array}{c|c|c} \left[\, e_1^3 \otimes e_1^{3T} \,\right] & \left[\, e_2^3 \otimes e_1^{2T} \,\right] & \left[\, e_3^3 \otimes e_1^{1T} \,\right] \\ \hline \left[\, e_2^3 \otimes e_2^{3T} \,\right] & \begin{bmatrix} e_1^2 \otimes e_1^{2T} \\ e_1^1 \otimes e_2^{2T} \end{bmatrix} & \left[\, O_{3\times 1} \,\right] \\ \hline \left[\, e_3^3 \otimes e_3^{3T} \,\right] & \left[\, e_2^3 \otimes e_2^{2T} \,\right] & \left[\, e_1^3 \otimes e_1^{1T} \,\right] \end{array}\right],$$

$$\text{(c)}\quad D_4^{*(d)} = \left[\begin{array}{c|c|c|c} \left[\, e_1^4 \otimes e_1^{4T} \,\right] & \left[\, e_2^4 \otimes e_1^{3T} \,\right] & \left[\, e_3^4 \otimes e_1^{2T} \,\right] & \left[\, e_4^4 \otimes e_1^{1T} \,\right] \\ \hline \left[\, e_2^4 \otimes e_2^{4T} \,\right] & \begin{bmatrix} e_1^2 \otimes e_1^{3T} \\ e_1^2 \otimes e_2^{3T} \end{bmatrix} & \left[\, e_4^4 \otimes e_2^{2T} \,\right] & \left[\, O_{4\times 1} \,\right] \\ \hline \left[\, e_3^4 \otimes e_3^{4T} \,\right] & \begin{bmatrix} e_2^3 \otimes e_2^{3T} \\ e_1^1 \otimes e_3^{3T} \end{bmatrix} & \begin{bmatrix} e_1^3 \otimes e_1^{2T} \\ O_{1\times 2} \end{bmatrix} & \left[\, O_{4\times 1} \,\right] \\ \hline \left[\, e_4^4 \otimes e_4^{4T} \,\right] & \left[\, e_3^4 \otimes e_3^{3T} \,\right] & \left[\, e_2^4 \otimes e_2^{2T} \,\right] & \left[\, e_1^4 \otimes e_1^{1T} \,\right] \end{array}\right].$$

Thus, we have a generally form of $D_n^{*(d)}$:

- for $i = 1$: $\left[\ \boldsymbol{e}_j^n \otimes \boldsymbol{e}_1^{n-(j-1)T}\ \right]$,
- for $j = 1$: $\left[\ \boldsymbol{e}_i^n \otimes \boldsymbol{e}_i^{nT}\ \right]$,
- for $i < j$ where $i = 2, ..., (n-1)$: $\begin{bmatrix} \boldsymbol{e}_{j+1}^n \otimes \boldsymbol{e}_i^{n-(j-1)T} \\ O_{n\times n-(j-1)} \end{bmatrix}$,
- for $i \geq j$ where $j = 2, 3, ..., n$: $\begin{bmatrix} \boldsymbol{e}_{i-(j-1)}^i \otimes \boldsymbol{e}_{i-(j-1)}^{n-(j-1)T} \\ \boldsymbol{e}_{j-1}^{n-i} \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_i^{n-(j-1)}T \\ O_{(n-i)\times n-(j-1)} \end{bmatrix}$.

Based on Definition 2.13 and Theorem 2.14, the Moore-Penrose inverse of $D_n^{*(d)}$ is given as

$$D_n^{*(d)+} = (D_n^{*(d)T} D_n^{*(d)})^{-1} D_n^{*(d)T} \tag{3.4}$$

Inspired by [12], we have the following relationships between $D_n^{*(d)}$ and $D_n^{*(d)+}$.

Theorem 3.6. *Let $A = [a_{ij}]$ be an $n \times n$ symmetric matrix, then*

a) $D_n^{*(h)} = D_n^{*(d)} B_n^{*(d)}$ *and* $D_n^{*(d)} = D_n^{*(h)} B_n^{*(d)}$,
b) $B_n^{*(d)} D_n^{*(d)+} = D_n^{*(h)+}$ *and* $B_n^{*(d)T} D_n^{*(h)+} = D_n^{*(d)+}$.

Proof.

a) Subtituting Equation 3.2 into 3.3, we have $vec(A) = D_n^{*(d)} B_n^{*(d)} vech^*(A)$. Next, subtituting $vec(A) = D_n^{*(h)} vech^*(A)$ into $vec(A) = D_n^{*(d)} B_n^{*(d)}$ $vech^*(A)$, we have $D_n^{*(h)} vech^*(A) = D_n^{*(d)} B_n^{*(d)} vech^*(A)$. So, it is implied that $D_n^{*(h)} = D_n^{*(d)} B_n^{*(d)}$ to the first result of part a). Since $B_n^{*(d)}$ is a non-singular matrix, then $B_n^{*(d)}$ has an inverse. The second result is obtained by post-multiplying both sides of $D_n^{*(h)} = D_n^{*(d)} B_n^{*(d)}$ by $B_n^{*(d)-1}$.

b) Subtituting $D_n^{*(h)} = D_n^{*(d)} B_n^{*(d)}$ into Equation 3.4, we have $D_n^{*(h)+} = B_n^{*(d)-1} (D_n^{*(d)T} D_n^{*(d)})^{-1} (B_n^{*(d)T})^{-1} B_n^{*(d)T} D_n^{*(d)T}$. Since $B_n^{*(d)}$ is an orthogonal matrix, that is $B_n^{*(d)-1} = B_n^{*(d)T}$, by Equation 3.4, the first result is obtained $B_n^{*(d)} D_n^{*(d)+} = D_n^{*(h)+}$. Next, substituting $D_n^{*(d)} = D_n^{*(h)} B_n^{*(d)}$ into Equation 3.4, we have $D_n^{*(d)+} = (B_n^{*(d)T})^T (D_n^{*(h)T} D_n^{*(h)})^{-1} D_n^{*(h)T}$, by Theorem 2.14, we have $B_n^{*(d)T} D_n^{*(h)+} = D_n^{*(d)+}$. □

4. Conclusions

This article provides new operator, i.e., *vech** and *vecd**. From these definitions, we have transformation matrix, which transforms *vech** to *vecd** operator for arbitary matrix $n \times n$ where $n \in N$. The transformed matrices $B_n^{*(d)}$ are permutation and orthogonal matrix, and the matrix has determinant 1 or -1. Then, it provides a marix analogous to the duplication matrix related to the $vecd^*$ and finds its properties.

5. Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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